JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 26, No. 5, September–October 2003

General Class of Tensegrity Structures: Topology and Prestress Equilibrium Analysis

Darrell Williamson
University of Wollongong, Wollongong, New South Wales 2522, Australia
and
Robert E. Skelton
University of California, San Diego, La Jolla, California 92093

A general class of tensegrity structures, consisting of both compression members, that is, bars, and tensile members, that is, cables, is defined. For a given number *N* of bars, we define the topological structure that is necessary to establish a tensegrity. Necessary and sufficient conditions for prestress mechanical equilibria of the tensegrity are then provided in terms of a nonlinear function of the position and orientation of the bars and the rest lengths of the cables.

I. Introduction

THE word "tensegrity" is a contraction of the words tension and integrity. A tensegrity structure was loosely defined by Fuller[1,2] as a "structural relationship in which structural shape is guaranteed by the interaction between a continuous network of members in tension and a set of members in compression." In Ref. 3, a more scientific definition is offered: "A tensegrity system is a stable connection of axially-loaded members. A class 1 tensegrity structure is one in which only one compressive member is connected to any node." The compressive members are required to achieve stability, and class 1 tensegrity structures have a continuous network of members in tension and a discontinuous network of members in compression. We shall refer to a compressive member as a bar and a tension member as a cable.

Tensegrity structures are mechanically stable because of the way in which the structures balance and distribute the mechanical stress and not principally as a result of the strength of the individual components. The bars that make up the frameworks are connected into particular configurations (triangles, pentagons, hexagons) and are oriented so that each joint is constrained to a fixed position. An example of a class 1 tensegrity is the Needle Tower sculpture of Kenneth Snelson that appears in the Krueller Mueller Museum, The Netherlands. This category of tensegrity structures was first invented by Snelson, who was a student of Fuller. (See Kenneth Snelson's November 1990 message to R. Moto on the origins of tensegrity at http://www.grunch.net/snelson/rmoto.html.) A class 2 tensegrity structure has more than one compressive member connected in a ball joint (so as not to apply torque from one member to another) to particular nodes.

Tensegrity structures have the property that, even before the application of any external load, members of the structures are already in compression or tension; that is, they are prestressed. The "rigid" bars sustain compressive forces, whereas the "elastic" cables sustain tensile forces when they "stretch" from their rest lengths. In fact, it is this prestress that stabilizes the tensegrity structure. Examples of tensegrity structures have been studied in mathematics, biology, and engineering. Connelly and Back[4] describe a class of tensegrity structures to be in terms of finite dimensional groups. Ingber[5,6] has used tensegrity structures to model cell biology. In engineering applications, tensegrity structures offer the possibility of considerable

Darrell Williamson is Director of the Telecommunications and Information Technology Research Institute and Professor of Engineering at the University of Wollongong, New South Wales, Australia. He has a B.E. and M.E. in electrical engineering and a B.Sc. in physics from the University of Newcastle, Australia, and a Ph.D. in applied mathematics from Harvard University. He has served as Dean of Engineering at the Australian National University and Chief Executive Officer of the Australian government-supported Cooperative Research Center in Advanced Computational Systems. His current research interests are in the modeling, analysis, and control of complex signals and systems, particularly in the areas of communication networks and multimedia. He has published two books on control and signal processing.

Robert E. Skelton earned a B.S. in electrical engineering from Clemson University, an M.E. in electrical engineering from the University of Alabama in Huntsville, and a Ph.D. from the University of California, Los Angeles, in mechanics and structures. He began his career with Lockheed Missiles and Space Company and then moved to Sperry Rand, designing controllers for spacecraft. He is Associate Editor of *Mathematical Modelling of Systems*, *Mathematical Problems in Engineering*, *Journal of Systems and Control*, and *Applied Mechanics Review*. He is a Fellow of AIAA and IEEE, a recipient of the Norman Medal in 2000, the 1991 Russell Severence Springer Professor at the University of California, Berkeley, a recipient of the Senior Scientist Award from the Alexander von Humboldt Foundation, and a recipient of a research award from the Japan Society for the Promotion of Science. He served on the National Research Council's Aeronautics and Engineering Board and on the External Independent Review Team for the servicing missions of the Hubble Space Telescope. He has published three books and over 150 journal papers. He holds patents on the control of tensegrity structures.

Received 30 March 2001; revision received 27 March 2002; accepted for publication 29 July 2002.

versatility if sensors and actuators are incorporated into the cables and bars. Furuya[7] and Moto[8] pointed out their versatility for space applications. More recently, Skelton et al.,[3] Skelton and Sultan,[9] and Sultan et al.[10] have analyzed the statics and dynamics of particular examples of the second category of tensegrity structures, which lays the foundation for further studies and applications.

This paper is concerned with the definition and analysis of a particular subclass of class 1 tensegrity structures, which we define as the $(N, S; P_1, P_2, \ldots, P_M)$ subclass. This subclass has N bars that form a discontinuous network and S cables that form a continuous network. The structure is based on the concept of a "stage" that, like the Needle Tower, grows in "height" with the number of stages. The defined subclass has M stages with P_k bars in the kth stage. Both symmetric structures (where the number of bars per stage is constant) and nonsymmetrical structures will be derived. Preliminary results[11,12] were reported earlier.

If an $(N, S; P_1, P_2, \ldots, P_M)$ tensegrity structure has N bars, its external geometry is characterized by the $2N$ coordinates of the "nodes," or "ends," of the bars. The resulting shape of the tensegrity then depends on 1) the properties of the constituent components (that is, on the lengths of the bars and the rest lengths of the cables), 2) the internal topology of how the bars and cables are connected (that is, on the unit vectors that define the orientation of the bars and cables), and 3) on the existence of self-stress or pretension (that is, on the stretched lengths of the cables) that is necessary to provide rigidity for the structure. Because it will be shown that material characteristics (in particular, Hookes' Law) are required to solve for internal forces, an $(N, S; P_1, P_2, \ldots, P_M)$ tensegrity is an example of an statically indeterminate structure.

In Sec. II, we begin by defining the topology of a one-stage (that is, $M = 1$) tensegrity and then proceed to develop the topology of multistage tensegrity structures. Structures are said to be symmetrical if the number of bars per stage is constant and are otherwise said to be nonsymmetrical. The topology of both symmetrical and nonsymmetrical structures are developed, and, in each case, the cable connections are specified. The end of each bar can possibly be connected by a cable to any one of the $2N - 2$ ends of the remaining $N - 1$ bars; that is, possible $2N - 2$ cables per node may be required. In comparison, it is shown that an $(N, S; P_1, P_2, \ldots, P_M)$ structure only requires between three and four cables per node.

In Sec. III, we develop necessary and sufficient conditions for prestress mechanical equilibrium in the absence of external applied forces. These conditions are expressed in terms of the solution of a system of algebraic equations of the form $\boldsymbol{Ap} = \boldsymbol{0}$ for a rectangular matrix $\boldsymbol{A}$. However, the determination of a solution is made more difficult because the components of $\boldsymbol{A}$ depend in a nonlinear way on the positions of the end points of the bars, and all components of the vector $\boldsymbol{p}$ must be positive. The paper finishes with some concluding remarks and directions for further research.

II. Topology

We now proceed to define a general $(N, S; P_1, P_2, \ldots, P_M)$ class of tensegrity structures consisting of N bars and S cables that are arranged into M stages with P_k bars in the kth stage such that

$$\sum_{k=1}^{M} P_k = N \tag{1}$$

An $(N = 2, S = 4; P_1 = 2)$ planar single- (that is, $M = 1$) stage structure is illustrated in Fig. 1. In this structure, the bar 12 is of length L_{12}, and the bar 34 is of length L_{34}. The cables, or tensile elements, are illustrated as springs.

As indicated in Fig. 1a, the structure is defined by first connecting cables [1, 3], [2, 3], and [2, 4]. Then a cable is connected from node 1 to node 2 with midpoint 4, and another cable is connected from node 3 to node 4 with midpoint 1. Finally, after connecting midpoint 1 to the end of stick 12 at node 1, and midpoint 4 to node 4, the closed tensegrity in Fig. 1b results. Note that, in the intermediate stage of Fig. 1a, there appears to be a total of seven cables. However, once node 1 and midpoint 1 and node 4 and midpoint 4 are made coincident, the number of cables reduces to $S = 4$. The resulting

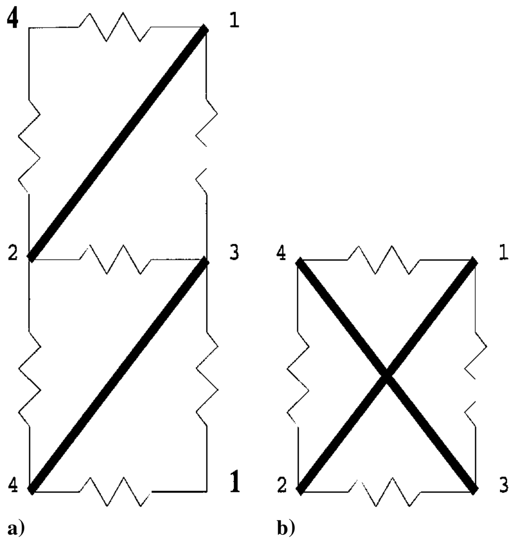

Fig. 1 Topology of (2, 4; 2) tensegrity.

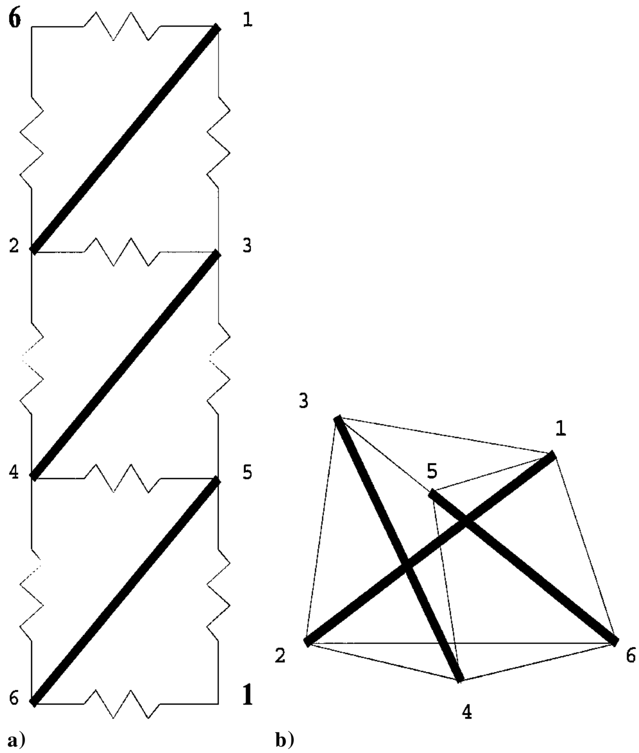

Fig. 2 Topology of (3, 9; 3) tensegrity.

structure is one stage (that is, $M = 1$) with $P_1 = N = 2$, that is, a (2, 4; 2) tensegrity.

For the (2, 4; 2) cable connection matrix, the resulting cable connections $[i, j]$ that connect node p_i to node p_j in the (2, 4; 2) structure are defined by

$$[1, 3], \qquad [1, 4], \qquad [2, 3], \qquad [2, 4]$$

Note that the connections $[2m - 1, 2m]$ for $m = 1, 2$ define the bars.

A. Single-Stage Structures

The planar (2, 4; 2) structure is a "trivial" $(N, S; P_1, P_2, \ldots, P_M)$ tensegrity in the sense that it has no "volume" and the bars "touch" (except in the mathematically ideal case where the bars have zero thickness). The simplest three-dimensional topology is provided by the one-stage $(N = 3, S = 9; P_1 = 3)$ tensegrity whose realization is illustrated in Fig. 2.

To begin, first (as illustrated in Fig. 2a) take bars 12, 34, and 56 and connect the cables [1, 3], [2, 3], [2, 4], [3, 5], [4, 5], and

[4, 6]. Then take another cable connecting node 5 to node 6 with midpoint 1, and connect midpoint 1 to 1. Similarly, connect the midpoint 6 of the cable that connects node 1 to node 2 to node 6. This realization results in the three-dimensional structure illustrated in Fig. 2b. Once again, as with the (2, 4; 2) structure, the number S of cables is reduced during the realization process. That is, in Fig. 2a, the structure first appears to have 10 cables, but the connection of node 1 to midpoint 1 and node 6 to midpoint 6 reduces this number to $S = 9$ cables. (The cables from node 6 to midpoint 1 and node 1 to midpoint 6 in Fig. 2a become identical in Fig. 2b.)

For the (3, 9; 3) cable connection matrix, cable connections $[i, j]$ that connect node p_i to node p_j in the (3, 9; 3) structure are defined by

$$[1, 3], \quad [1, 5], \quad [1, 6], \quad [2, 3], \quad [2, 4], \quad [2, 6]$$

$$[3, 5], \quad [4, 5], \quad [4, 6]$$

Note that the connections $[2m - 1, 2m]$ for $m = 1, 2, 3$ define the bars.

Next, topological equivalence is explained, as follows. The actual coordinates of the nodes of the bars will depend on the lengths $\{L_{2m-1,2m};\ m = 1, 2\}$ of the bars and the rest lengths $\{\ell^0_{pq}\}$ of the cables. Consequently, an equivalence class of tensegrity structures exists. Specifically, one $(N, S;\ P_1, P_2, \ldots, P_M)$ structure is said to be topologically equivalent to another structure if the first can be derived from the second by means of either a continuous change in the length of one or more bars, or a continuous change in the rest length of one or more cables subject to the condition that all cable tensions remain nonnegative and no bars remain in contact, that is, no two bars share a common point.

Single-stage $(M = 1)$ three-dimensional structures $(N, S;\ P_1 = N)$ for any value of $N \geq 3$ may be similarly realized. The distinguishing feature of this equivalence class of tensegrity structures is that both the base and the top form an N-sided polygon. For $N = 3$, the base (top) as defined by the bar ends $\{2, 4, 6\}$, and the top (base) as defined by the bar ends $\{1, 3, 5\}$ form triangles. As illustrated Fig. 3, the base (top) of the general N bar structure is formed by the bar ends $\{2, 4, 6, \ldots, 2N\}$, whereas the top (base) is formed by the bar ends $\{1, 3, 5, \ldots, 2N - 1\}$. Also note that after initially beginning with $3N + 1$ cables, the three-dimensional closure that occurs when midpoint $2N$ is connected to node $2N$, and midpoint 1 is connected to node 1, results in only $3N$ cables. We summarize this result as follows.

Theorem 1: The one-stage $(M = 1)$ N bar tensegrity has a $(N, S = 3N;\ P_1 = N)$ structure. The bar ends that define both the base and top form an N-sided polygon. The $3N$ cable connections are defined by the connection matrix:

$$[1, 2N - 1], \qquad [1, 2N], \qquad [2, 2N]$$

For $1 \leq m \leq N - 1$

$$[2m + 1, 2m - 1], \qquad [2m + 1, 2m], \qquad [2m + 2, 2m]$$

Note that the connections $[2m - 1, 2m]$ for $1 \leq m \leq N$ define the bars.

As with a (2, 4; 2) tensegrity, the actual coordinates of the nodes of the bars of an $(N, 3N;\ N)$ tensegrity structure will depend on the lengths $\{L_{2m-1,2m};\ 1 \leq m \leq N\}$ of the N bars and the rest lengths of the cables $[p, q]$. Once again, an equivalence class of tensegrity structures can be defined by means of a continuous change in the length of the bars or the rest lengths of the cables, subject to the conditions that all cable tensions remain positive and no bars remain in contact.

B. Two-Stage Structures

The simplest two-stage (that is, $M = 2$) structure is the $(N = 3,\ S = 9;\ P_1 = 2,\ P_2 = 1)$ tensegrity illustrated in Fig. 4. Stage 1 consists of bars 12 and 34, whereas stage 2 consists of bar 56. The realization begins (as illustrated in Fig. 4a) by connecting cables [1, 5], [2, 5], [3, 5], and [3, 6]. Then, similar to the realization of single-stage structures, midpoint 1 is connected to node 1, midpoint 2 is connected to node 2, midpoint 4 is connected to node 4, and midpoint 6 is connected to node 6, resulting in a three-dimensional structure. Initially, in Fig. 3, there appear to be 12 cables that reduce to 9 after the connections of the various midpoints.

For the (3, 9; 2, 1) cable connection matrix, cable connections $[i, j]$ that connect node p_i to node p_j are defined by

$$[1, 4], \quad [1, 5], \quad [1, 6], \quad [2, 3], \quad [2, 4], \quad [2, 5]$$

$$[3, 5], \quad [3, 6], \quad [4, 6]$$

When compared with the single-stage (3, 9; 3) structure (which also has $N = 3$ bars and $S = 9$ cables), the (3, 9; 2, 1) structure can be demonstrated after realization (by sticks and soft elastic bands) to have a greater clockwise stiffness, in that if a clockwise torque is applied to the top while the base is held fixed, then there is more resistance from the (3, 9; 2, 1) structure than for the (3, 9; 3) structure. That is, the different topologies of the (3, 9; 3) and the (3, 9; 2, 1) tensegrity structures result in different mechanical properties. [This is not evident from the connection matrices of the two structures, and the stiffness properties of $(N, S;\ P_1, P_2, \ldots, P_M)$ tensegrities are not the subject of this paper.]

The two-stage structures $(N = 4,\ S = 14;\ P_1 = 2,\ P_2 = 2)$, $(N = 5,\ S = 18;\ P_1 = 3,\ P_2 = 2)$, and $(N = 6,\ S = 24,\ P_1 = 3,\ P_2 = 3)$ are

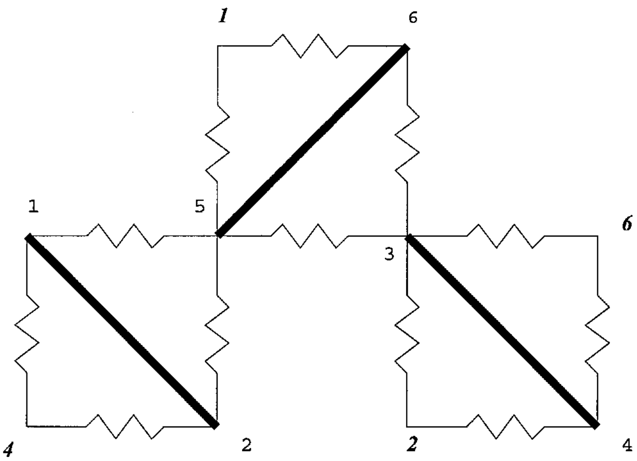

Fig. 4 Planar topology of (3, 9; 2, 1) tensegrity.

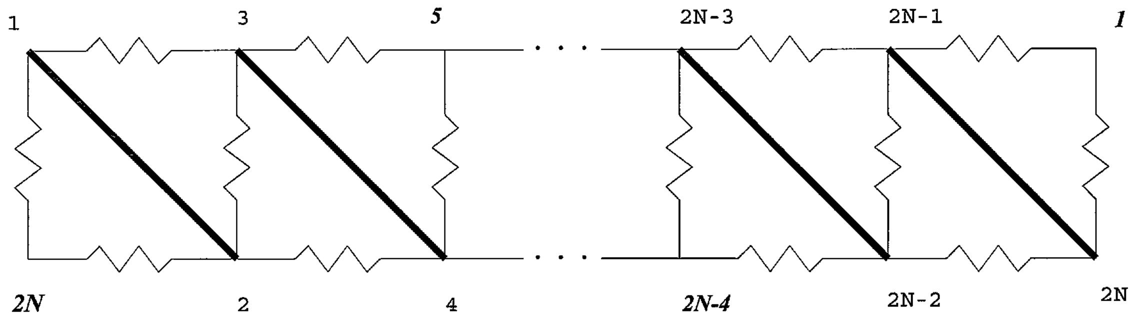

Fig. 3 Planar topology of $(N, 3N;\ N)$ tensegrity.

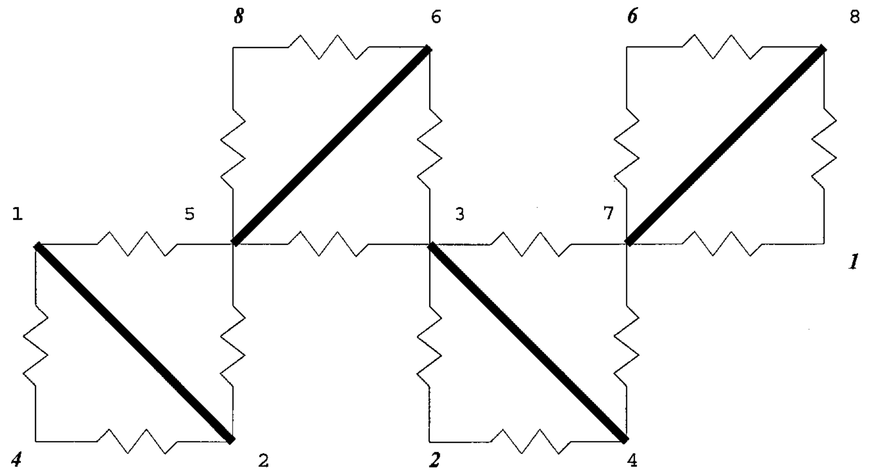

Fig. 5 Planar topology of (4, 14; 2, 2) tensegrity.

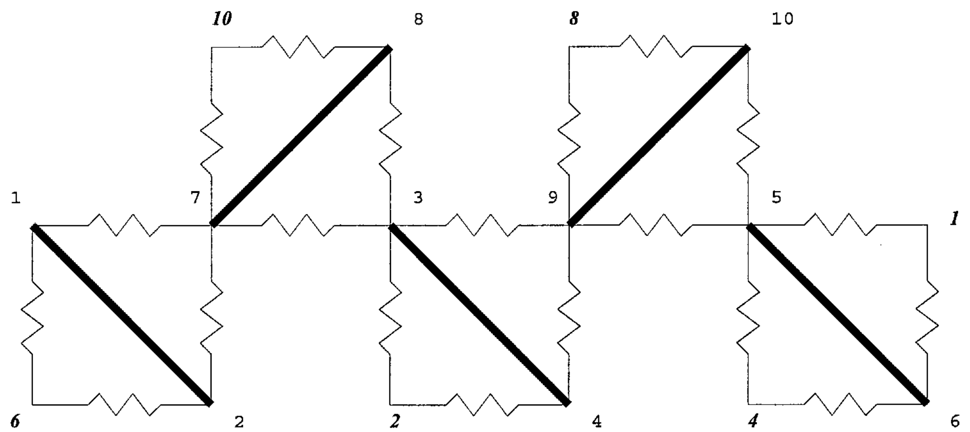

Fig. 6 Planar topology of (5, 18; 3, 2) tensegrity.

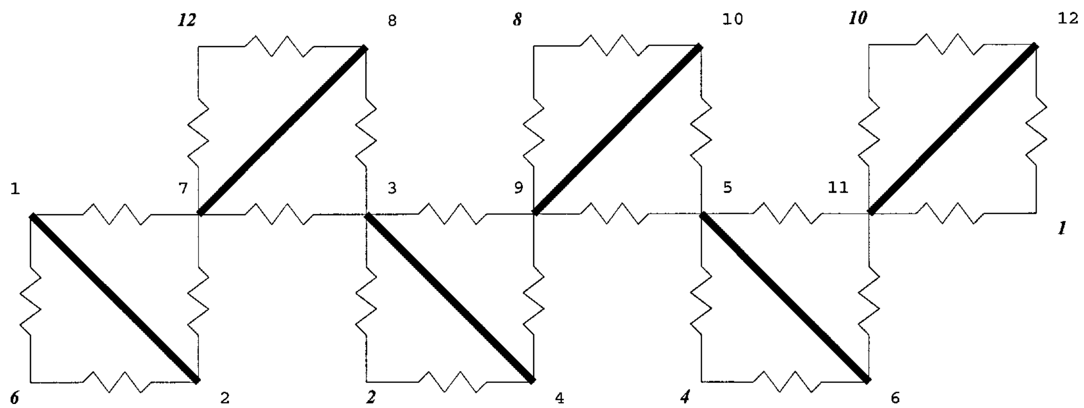

Fig. 7 Planar topology of (6, 24; 3, 3) tensegrity.

illustrated in Figs. 5–7, respectively. In the (4, 14; 2, 2) structure, the $4 \times 4 = 16$ initial cables are reduced to $S = 14$ cables after connection of midpoint 1 to node 1, midpoint 2 to node 2, midpoint 4 to node 4, midpoint 6 to node 6, and midpoint 8 to node 8. Similarly, the (5, 18; 3, 2) structure is reduced from $5 \times 4 = 20$ to $S = 18$ cables, whereas the (6, 24; 3, 3) requires the full $S = 6 \times 4 = 24$ cables.

C. *M*-Stage Symmetrical Structures

Both the two-stage (4; 14; 2, 2) and the two-stage (6, 24; 3, 3) structures are examples of symmetrical structures, in that in both cases $P_1 = P_2$. More generally, an $(N, S; P_1, P_2, \ldots, P_M)$ tensegrity structure is said to be symmetrical if it has the same number of bars per stage, that is, for all k,

$$P_k = P$$

It follows that a symmetrical structure has $N = MP$ bars. As described later, a symmetrical $(N, S; P_1, P_2, \ldots, P_M)$ structure (except in the trivial case when $M = 1$) has the maximum number of four cables per bar.

Theorem 2: A symmetrical $(N = MP, S; P, P, \ldots, P)$ tensegrity structure with $M > 1$ stages has S cables where

$$S = \begin{cases} 4N - 2, & P = 2 \\ 4N, & P \geq 3 \end{cases}$$

The cable connections for the kth stage where $1 \leq k \leq M$ are defined as follows.

For $k = 1$; $P = 2$,

$$[1, 4], \quad [1, 5], \quad [1, 7], \quad [1, 8], \quad [2, 3], \quad [2, 4]$$

$$[2, 5], \quad [3, 5], \quad [3, 6], \quad [3, 7]; \quad [4, 7]$$

and for $k = 1$; $P \geq 3$,

$$[1, 2P], \quad [1, 2P+1], \quad [1, 4P-1], \quad [1, 4P]$$

$$[2, 3], \quad [2, 4], \quad [2, 2P], \quad [2, 2P+1]$$

For $1 \leq m \leq P-2$,

$$[2m+1, 2P+2m-1], \quad [2m+1, 2P+2m]$$

$$[2m+1, 2P+2m+1], \quad [2m+2, 2m+3]$$

$$[2m+2, 2m+4], \quad [2m+2, 2P+2m+1]$$

$$[2P-1, 4P-3], \quad [2P-1, 4P-2]$$

$$[2P-1, 4P-1], \quad [2P, 4P-1]$$

For $k = 2n < M$; $0 \leq m \leq P-2$,

$$[(4n-2)P+2m+1, 4nP+2m+2]$$

$$[(4n-2)P+2m+2, 4nP+2m+2]$$

$$[(4n-2)P+2m+2, 4nP+2m+3]$$

$$[(4n-2)P+2m+2, 4nP+2m+4]$$

$$[4nP-1, (4n+2)P], \quad [4nP, 4nP+1]$$

$$[4nP, 4nP+2], \quad [4nP, (4n+2)P]$$

For $k = 2n+1 < M$,

$$[4nP+1, (4n+2)P+1], \quad [4nP+1, (6n+2)P-1]$$

$$[4nP+1, (6n+2)P], \quad [4nP+2, (4n+2)P+1]$$

For $1 \leq m \leq P-1$,

$$[4nP+2m+1, (4n+2)P+2m-1]$$

$$[4nP+2m+1, (4n+2)P+2m]$$

$$[4nP+2m+1, (4n+2)P+2m+1]$$

$$[4nP+2m+2, (4n+2)P+2m+1]$$

For $k = 2n = M$; $P = 2$,

$$[8n-3, 8n], \quad [8n-2, 8n-1], \quad [8n-2, 8n]$$

and for $k = 2n = M$; $P \geq 3$,

$$[(4n-2)P+1, 4nP], \quad [(4n-2)P+2, 4nP]$$

For $1 \leq m \leq P-1$,

$$[(4n-2)P+2m, (4n-2)P+2m+1]$$

$$[(4n-2)P+2m, (4n-2)P+2m+2]$$

For $k = 2n+1 = M$; $P = 2$

$$[8n+1, 8n+3], \quad [8n+1, 8n+4], \quad [8n+2, 8n+3]$$

and for $k = 2n+1 = M$; $P \geq 3$,

$$[4nP+1, 4nP+3], \quad [4nP+1, (4n+2)P-1]$$

$$[4nP+1, (4n+2)P], \quad [4nP+2, 4nP+3]$$

For $1 \leq m \leq P-2$,

$$[4nP+2m+1, 4nP+2m+3]$$

$$[4nP+2m+2, 4nP+2m+3]$$

D. *M*-Stage Nonsymmetrical Structures

A nonsymmetrical $(N, S; P_1, P_2, \ldots, P_M)$ tensegrity structure is one for which $P_k \neq P_\ell$ for some k and ℓ. Nonsymmetrical structures of N bars have less than $4N$ cables. For example, the (5, 18; 3, 2) structure in Fig. 6 has $S = 18 < 20(=4N)$ cables. The following result gives both the precise number of cables for each structure and provides conditions for the allowable number of bars P_k in stage k.

Theorem 3: An $(N, S; P_1, P_2, \ldots, P_M)$ tensegrity structure exists when the following two conditions are satisfied: First,

$$P_1 > 1, \qquad P_k \geq 1 \quad \text{for} \quad k \geq 2$$

and when $P_r = 1$, then $M = r$. Second,

$$|P_{k+1} - P_k| = 1 \quad \text{for} \quad 1 \leq k \leq M-1$$

The number S of cables for all $(N, S; P_1, P_2, \ldots, P_M)$ structures is bounded according to

$$3N \leq S \leq 4N$$

In particular, for symmetrical structures,

$$S = \begin{cases} 3N, & M = 1, \quad P \geq 3 \\ 4N-2, & M \geq 2, \quad P = 2 \\ 4N, & M \geq 2, \quad P \geq 3 \end{cases}$$

and for nonsymmetrical structures (and so $M \geq 2$)

$$S = 4N - L, \qquad L = \sum_{k=1}^{M} \ell_k$$

where

$$\ell_1 = 1 \quad \text{if} \quad P_1 = 2 \quad \text{or} \quad P_2 = P_1 - 1$$

$$\ell_k = 1 \quad \text{for} \quad 2 \leq k \leq M-1$$

$$\text{if} \quad P_k = P_{k-1} + 1 \quad \text{and} \quad P_{k+1} = P_{k-1}$$

$$\ell_M = \begin{cases} 1 & \text{if} \quad P_M = 2 \quad \text{or} \quad P_M = P_{M-1} + 1 \\ 2 & \text{if} \quad P_M = 1 \end{cases}$$

An $(N, S; P_1, P_2, \ldots, P_M)$ structure is equivalent to an $(N, S; Q_1, Q_2, \ldots, Q_M)$ structure when

$$P_k = Q_{M+1-k}, \qquad 1 \leq k \leq M$$

The case when $S = 3N$ is provided in Theorem 1, and the other symmetrical cases are provided in Theorem 2. [A symmetrical (6, 24; 3, 3) structure is illustrated in Fig. 7.] The proof of the result for nonsymmetrical structures is by inspection and realization. The situation when $\ell_1 = 1$ in Theorem 2 is illustrated by the (3, 9; 2, 1) structure in Fig. 4, the (4, 14; 2, 2) structure in Fig. 5, and the (5, 18; 3, 2) structure in Fig. 6. The situation when $\ell_k = 1$ for some $2 \leq k \leq M-1$ is demonstrated by the (7, 25; 2, 3, 2) structure illustrated in Fig. 8. The situation when $\ell_M = 1$ is demonstrated by the (4, 14; 2, 2) structure and the (5, 18; 2, 3) structure [which is topologically equivalent to the (5, 18; 3, 2) structure] in Fig. 5. The situation when $\ell_M = 2$ is demonstrated by the (3, 9; 2, 1) structure.

For a given number N of bars, Theorem 3 may be used to determine all possible $(N, S; P_1, P_2, \ldots, P_M)$ tensegrity structures. For $N = 3$, the only structures are (3, 9; 3) and (3, 9; 2, 1). For the nonsymmetrical structure (3, 9; 2, 1), we have, from Theorem 3, that $\ell_1 = 1$ and $\ell_2 = 2$, and so $S = 4 \times 3 - 3 = 9$. All possible structures for $3 \leq N \leq 10$ are provided in Table 1. Note that even though any number of additional cables may be added from any node to any other node, such additional cables are unnecessary to maintain stable equilibrium of the structure.

Two further ways in which Theorem 3 can be applied are detailed as follows:

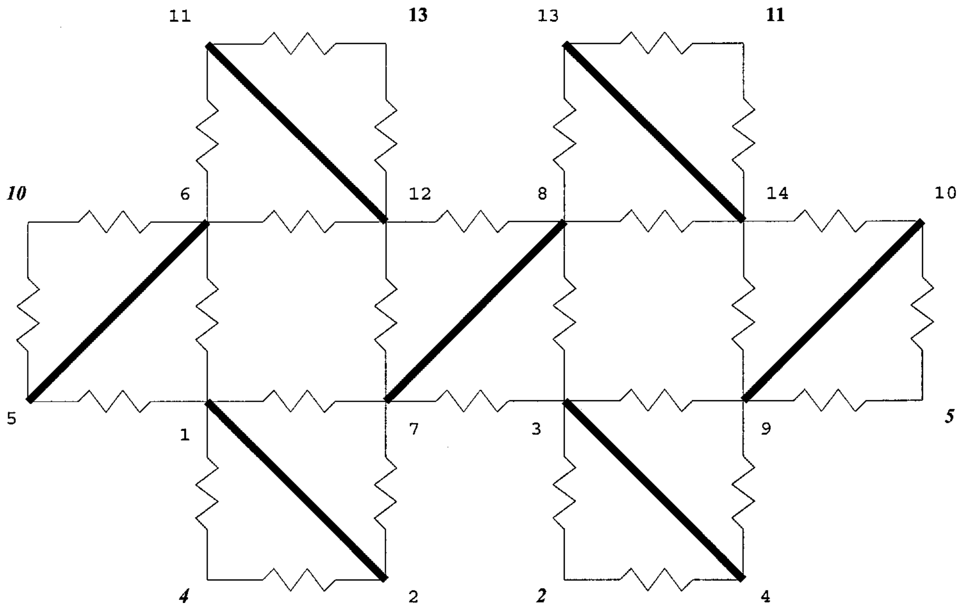

Fig. 8 Planar topology of (7, 25; 2, 3, 2) tensegrity.

Table 1 Tensegrity structures with $3 \le N \le 10$ bars

Number of struts	Structures
3	(3, 9; 3), (3, 9; 2, 1)
4	(4, 12; 4), (4, 14; 2, 2)
5	(5, 15; 5), (5, 18; 3, 2)
6	(6, 18; 6), (6, 24; 3, 3), (6, 21; 3, 2, 1), (6, 22; 2, 2, 2)
7	(7, 21; 7), (7, 27; 4, 3), (7, 26; 3, 2, 2), (7; 24; 2, 2, 2, 1), (7, 26; 2, 3, 2)
8	(8, 24; 8), (8, 32; 4, 4), (8, 31; 3, 3, 2), (8, 31; 3, 2, 3)
9	(9, 27; 9), (9, 35; 5, 4), (9, 35; 4, 3, 2), (9, 36; 3,3, 3), (9, 33; 3, 2, 1)
10	(10, 30; 10), (10, 40; 5, 5), (10, 39; 4, 3, 3), (10, 38; 4, 3, 2, 1), (10, 39; 3, 4, 3), (10, 38; 3, 2, 2, 3)

1) Consider the $M = 10$ stage (33, S; 3, 2, 3, 4, 5, 4, 3, 2, 3, 4) structure. Then

$$\ell_1 = \ell_5 = \ell_{10} = 1, \qquad \text{otherwise} \qquad \ell_k = 0$$

Hence, $L = 3$, and so $S = 4N - L = 129$.

2) Consider the $M = 9$ stage (22, S; 3, 2, 2, 2, 3, 4, 3, 2, 1) structure. Then

$$\ell_1 = \ell_6 = 1, \qquad \ell_9 = 2, \qquad \text{otherwise} \qquad \ell_k = 0$$

Hence, $L = 4$, and so $S = 4N - L = 84$.

As for the case of a symmetrical tensegrity structure, a recursive algorithm can also be developed to define the cable connections of an M-stage nonsymmetrical tensegrity structure.

III. Prestress Equilibrium: Nonlinear Algebraic Conditions

A tensegrity structure is said to be in mechanical equilibrium if all geometric constraints are consistent and all bars are in force equilibrium. In this section, we derive necessary and sufficient conditions on the positions $\{(x_k, y_k, z_k);\ 1 \le k \le 2N\}$ of the bar endpoints (or nodes) so that an $(N, S; P_1, P_2, \ldots, P_M)$ tensegrity structure is in mechanical equilibrium.

From the preceding section on topology, all nodes are defined by the ends of bars. In particular, the mth bar is defined by the two nodes $\{2m-1, 2m\}$. We say that the mth bar is strongly associated with the two (neighboring) nodes $\{p_m, q_m\}$ if, in the "planar topology," the four nodes $\{2m-1, p, 2m, q\}$ define a "square." For example, in the (2, 4; 2) planar structure in Fig. 1a, bar 12 is strongly associated with nodes {3, 4} and bar 34 is strongly associated with nodes {1, 2}. In the $(N, 3N; N)$ planar structure in Fig. 3, the mth bar for $2 \le m \le N-1$ is strongly associated with nodes $\{2m-2, 2m+1\}$, whereas the first bar (that is, bar 12) is strongly associated with nodes $\{2N, 3\}$, and the Nth bar is strongly associated with nodes $\{2N-2, 1\}$. The two nodes that are strongly associated with each bar in multistage (that is, $M > 1$) structures, such as those in Figs. 7 and 8, are similarly defined. Based on this definition, we have the following result.

Lemma 1: Consider an $(N, S; P_1, P_2, \ldots, P_M)$ tensegrity structure in which the mth bar of length $L_{2m-1,2m}$ is defined by nodes $\{2m-1, 2m\}$ and is strongly associated with nodes $\{p_m, q_m\}$. Suppose $\{\boldsymbol{t}_{2m-1}, \boldsymbol{t}_{2m}\}$ are the resultant forces at nodes $2m-1$ and $2m$, respectively. Also, suppose $\ell_{pq} > 0$ is the length of the cable connecting node p to node q (when one exists), and $\boldsymbol{e}_{rs}$ is the unit vector directed from node r to node s. Then the following conditions apply.

1) Geometrical consistency:

$$\ell_{p_m,2m}\boldsymbol{e}_{p_m,2m} - \ell_{p_m,2m-1}\boldsymbol{e}_{p_m,2m-1} = L_{2m-1,2m}\boldsymbol{e}_{2m-1,2m}$$
$$\ell_{q_m,2m}\boldsymbol{e}_{q_m,2m} - \ell_{q_m,2m-1}\boldsymbol{e}_{q_m,2m-1} = L_{2m-1,2m}\boldsymbol{e}_{2m-1,2m} \tag{2}$$

2) Force equilibrium in the absence of externally applied forces:

$$\boldsymbol{t}_{2m-1} = \gamma_{2m-1}\boldsymbol{e}_{2m-1,2m}, \qquad \boldsymbol{t}_{2m} = -\boldsymbol{t}_{2m-1} \tag{3}$$

or, equivalently,

$$\boldsymbol{H}_{2m-1,2m}\boldsymbol{t}_{2m-1} = \boldsymbol{0}, \qquad \boldsymbol{t}_{2m} = -\boldsymbol{t}_{2m-1} \tag{4}$$

where (with $\boldsymbol{I}$ the identity matrix)

$$\boldsymbol{H}_{2m-1,2m} = \boldsymbol{I} - \boldsymbol{e}_{2m-1,2m}\boldsymbol{e}^T_{2m-1,2m}, \qquad \gamma_{2m-1} = \boldsymbol{e}^T_{2m-1,2m}\boldsymbol{t}_{2m-1} \tag{5}$$

The conditions for geometrical consistency follow directly from the geometry of the planar topology. To establish the condition for force equilibrium, observe that in the absence of external forces, the forces on each bar are only applied at two nodes of the bar. Consequently, the force system on each bar defines a two-force problem, which implies that both the resultant force $\boldsymbol{t}_{2m-1}$ at node $2m-1$, and the resultant force $\boldsymbol{t}_{2m}$ at node $2m$, are directed along the mth bar. Furthermore, because each bar is in force equilibrium, $\boldsymbol{t}_{2m-1} + \boldsymbol{t}_{2m} = \boldsymbol{0}$, which gives the result in Eq. (3). After taking the inner product of both sides of Eq. (3) with $\boldsymbol{e}_{2m-1,2m}$, the expressions for γ_{2m-1} and $\boldsymbol{H}_{2m-1,2m}$ in Eqs. (4) and (5) follow.

The tension $\boldsymbol{t}_{pq}$ in the cable $[p, q]$ connecting node p to node q can be expressed in the form

$$\boldsymbol{t}_{pq} = \alpha_{pq}\boldsymbol{e}_{pq}, \qquad \alpha_{pq} \geq 0 \tag{6}$$

where α_{pq} is the magnitude of the force (or cable force coefficient), and $\boldsymbol{e}_{pq}$ is a unit vector directed from node p to node q. Then, assuming a linear force relationship in the cables, we have, from Hookes' Law, that α_{pq} in Eq. (6) is given by

$$\alpha_{pq} = k_{pq}\left(\ell_{pq} - \ell^0_{pq}\right), \qquad \ell_{pq} \geq \ell^0_{pq} \geq 0 \tag{7}$$

where $k_{pq} > 0$ is the spring constant of the cable $[p, q]$, and ℓ^0_{pq} is the rest length of cable $[p, q]$. Because $\alpha_{pq} \geq 0$, the rest length ℓ^0_{pq} must not be greater than the actual length ℓ_{pq}. (Mathematically ideal cables can have zero rest length and/or zero force.)

The force equilibrium conditions in Lemma 1 are only given in terms of resultant forces. A complete description requires these resultant forces to be expressed in terms of the unit cable vectors $\boldsymbol{e}_{rs}$ and cable force coefficients $\alpha_{rs} \geq 0$. We now proceed with this step for the different tensegrity topologies. For this purpose, we adopt the convention that the (vector) tension $\boldsymbol{t}_{k\ell}$ in the cable connecting node k to node ℓ is in the direction from node k to node ℓ, which implies

$$\boldsymbol{t}_{k\ell} = -\boldsymbol{t}_{\ell k} \tag{8}$$

Consequently, when writing down the force equilibrium equations, we only need to use cable tensions $\boldsymbol{t}_{pq}$ for $p < q$.

A. Planar $N = 2$ Bar Tensegrity

Consider two possible (2, 4; 2) structures defined in the xy plane where node k has coordinates (x_k, y_k), with the corresponding force diagrams as illustrated in Fig. 9. We assume that the bars are ideal, with zero thickness (they do not touch) so that the only forces that are exerted on the system occur at nodes 1, 2, 3, and 4. Because the force $\boldsymbol{t}_1 = \boldsymbol{t}_{13} + \boldsymbol{t}_{14}$ in Fig. 9b is not directed along bar 12, we conclude that the topology in Fig. 9b is not possible. An equilibrium configuration is only possible when the two bars cross over each other (as in Fig. 9a).

1. *Geometrical Consistency*

From Fig. 9, the geometry is determined by the four unit vectors $\{\boldsymbol{e}_{13}, \boldsymbol{e}_{14}, \boldsymbol{e}_{23}, \boldsymbol{e}_{24}\}$, the cable lengths $\{\ell_{13}, \ell_{14}, \ell_{23}, \ell_{24}\}$, and the bar lengths $\{L_{12}, L_{34}\}$ such that

$$\ell_{13}\boldsymbol{e}_{13} - \ell_{23}\boldsymbol{e}_{23} = L_{12}\boldsymbol{e}_{12}, \qquad \ell_{14}\boldsymbol{e}_{14} - \ell_{24}\boldsymbol{e}_{24} = L_{12}\boldsymbol{e}_{12}$$
$$-\ell_{13}\boldsymbol{e}_{13} + \ell_{14}\boldsymbol{e}_{14} = L_{34}\boldsymbol{e}_{34}, \qquad -\ell_{23}\boldsymbol{e}_{23} + \ell_{24}\boldsymbol{e}_{24} = L_{34}\boldsymbol{e}_{34} \tag{9}$$

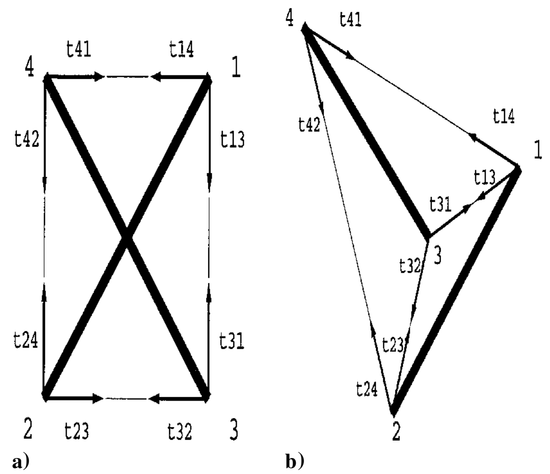

Fig. 9 Force diagram for (2, 4; 2) tensegrity.

where the unit vector $\boldsymbol{e}_{pq}$ and the length ℓ_{pq} are given by

$$\boldsymbol{e}_{pq} \triangleq \frac{1}{\ell_{pq}}\begin{bmatrix} x_p - x_q \\ y_p - y_q \end{bmatrix}, \qquad \ell_{pq} \triangleq \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2} \tag{10}$$

Now, from the first equation involving L_{34} and the two equations involving L_{12}

$$L_{34}\boldsymbol{e}_{34} = -\{L_{12}\boldsymbol{e}_{12} + \ell_{23}\boldsymbol{e}_{23}\} + \{L_{12}\boldsymbol{e}_{12} + \ell_{24}\boldsymbol{e}_{24}\}$$
$$= -\ell_{23}\boldsymbol{e}_{23} + \ell_{24}\boldsymbol{e}_{24}$$

which is the last equation for L_{34} in Eqs. (9). That is, there are actually only three independent geometric constraints in Eqs. (9). It is clear in this planar case that, for any four vectors $\{\boldsymbol{e}_{13}, \boldsymbol{e}_{14}, \boldsymbol{e}_{23}, \boldsymbol{e}_{24}\}$, of which at most two are parallel to one another, there exist cable and bar lengths $\{\ell_{ij} > 0, L_{mn} > 0\}$ that enable the geometrical consistency constraints (9) to be satisfied.

2. *Force Equilibrium*

From Lemma 1, the force balance equation on bar 12 is

$$\boldsymbol{H}_{12}\boldsymbol{t}_1 = \boldsymbol{0}, \qquad \boldsymbol{t}_1 + \boldsymbol{t}_2 = \boldsymbol{0} \tag{11}$$

where, from Fig. 9, the resultant force $\boldsymbol{t}_1(\boldsymbol{t}_2)$ applied at node 1 (node 2) is given by

$$\boldsymbol{t}_1 = \boldsymbol{t}_{13} + \boldsymbol{t}_{14}, \qquad \boldsymbol{t}_2 = \boldsymbol{t}_{23} + \boldsymbol{t}_{24} \tag{12}$$

Likewise, the force balance equation on bar 34 is

$$\boldsymbol{H}_{34}\boldsymbol{t}_3 = \boldsymbol{0}, \qquad \boldsymbol{t}_3 + \boldsymbol{t}_4 = \boldsymbol{0} \tag{13}$$

where the resultant force $\boldsymbol{t}_3(\boldsymbol{t}_4)$ applied at node 3 (node 4) using Eq. (8) is given by

$$\boldsymbol{t}_3 = \boldsymbol{t}_{31} + \boldsymbol{t}_{32} = -\boldsymbol{t}_{13} - \boldsymbol{t}_{23}, \qquad \boldsymbol{t}_4 = \boldsymbol{t}_{41} + \boldsymbol{t}_{42} = -\boldsymbol{t}_{14} - \boldsymbol{t}_{24} \tag{14}$$

Note that $\boldsymbol{t}_1 + \boldsymbol{t}_2 = \boldsymbol{0}$ implies $\boldsymbol{t}_3 + \boldsymbol{t}_4 = \boldsymbol{0}$, and so there are actually only three independent force equilibrium equations in Eqs. (11) and (13).

Definition 1: Given any two p-dimensional vectors $\{\boldsymbol{v}, \boldsymbol{w}\}$ with their kth components given by v_k and w_k, respectively, we say that

$$\boldsymbol{0} < \boldsymbol{v} < \boldsymbol{w} \tag{15}$$

if $0 < v_k < w_k$ for all $1 \leq k \leq p$.

Clearly, cables with zero force coefficient can either be added to, or subtracted from, an equilibrium configuration without effecting the equilibrium state. We, therefore now seek only equilibrium solutions in which all force coefficients are strictly positive. From Lemma 1, we have the following result.

Theorem 4: Consider a (2, 4; 2) tensegrity structure with nodes 1, 2, 3, and 4, as illustrated in Fig. 9, in which node k has coordinates (x_k, y_k). Define the unit vectors $\boldsymbol{e}_{pq}$ by Eq. (10), and the cable vector $\boldsymbol{l}$ and the cable coefficient vector $\boldsymbol{p}$ by

$$\boldsymbol{l}^T = [\ell_{13} \quad \ell_{14} \quad \ell_{23} \quad \ell_{24}], \qquad \boldsymbol{p}^T = [\alpha_{13} \quad \alpha_{14} \quad \alpha_{23} \quad \alpha_{24}] \tag{16}$$

Then the following hold:

1) The geometry determined by the six unit vectors $\{\boldsymbol{e}_{12}, \boldsymbol{e}_{13}, \boldsymbol{e}_{14}, \boldsymbol{e}_{23}, \boldsymbol{e}_{24}, \boldsymbol{e}_{34}\}$ is consistent if there exists a solution $\boldsymbol{l}$ of the equation

$$\boldsymbol{G}\boldsymbol{l} = \boldsymbol{b}, \qquad \boldsymbol{l} > \boldsymbol{0} \tag{17}$$

where the 6×4 matrix $\boldsymbol{G}$, and the six-vector $\boldsymbol{b}$ are given by

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{e}_{13} & \boldsymbol{0} & -\boldsymbol{e}_{23} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{e}_{14} & \boldsymbol{0} & -\boldsymbol{e}_{24} \\ -\boldsymbol{e}_{13} & \boldsymbol{e}_{14} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} L_{12}\boldsymbol{e}_{12} \\ L_{12}\boldsymbol{e}_{12} \\ L_{34}\boldsymbol{e}_{34} \end{bmatrix} \tag{18}$$

2) The consistent geometry defined by Eqs. (16–18) is in force equilibrium if and only if there exists a solution $\boldsymbol{p}$ of the equation

$$\boldsymbol{A}\boldsymbol{p} = \boldsymbol{0}, \qquad \boldsymbol{p} > \boldsymbol{0} \tag{19}$$

where the 6×4 matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{H}_{12}\boldsymbol{e}_{13} & \boldsymbol{H}_{12}\boldsymbol{e}_{14} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{e}_{13} & \boldsymbol{e}_{14} & \boldsymbol{e}_{23} & \boldsymbol{e}_{24} \\ \boldsymbol{H}_{34}\boldsymbol{e}_{13} & \boldsymbol{0} & \boldsymbol{H}_{34}\boldsymbol{e}_{23} & \boldsymbol{0} \end{bmatrix},$$

$$\boldsymbol{H}_{12} = \boldsymbol{I} - \boldsymbol{e}_{12}\boldsymbol{e}_{12}^T, \qquad \boldsymbol{H}_{34} = \boldsymbol{I} - \boldsymbol{e}_{34}\boldsymbol{e}_{34}^T \tag{20}$$

The proof for part 1 follows directly from the geometry. To prove part 2, assume there is a solution $\boldsymbol{l} > \boldsymbol{0}$ of Eq. (17). Suppose Eq. (19) has a solution $\boldsymbol{p} > \boldsymbol{0}$. Then force equilibrium is established with force coefficient vector $\boldsymbol{p}$, and given $\{\alpha_{pq} > 0, \ell_{pq} > 0\}$, a cable rest length ℓ_{pq}^0 can always be chosen to satisfy Eq. (7). If any component α_{pq} of $\boldsymbol{p}$ is negative, then $\ell_{pq} > 0$ implies that $\ell_{pq}^0 > \ell_{pq}$, which is impossible. This completes the proof. Note that if $\boldsymbol{p}_1 > \boldsymbol{0}$ is a solution of Eq. (19), then so too is $\beta\boldsymbol{p}_1 > \boldsymbol{0}$ for any $\beta > 0$. Scaling all of the cable tensions (by adjusting β) does not change the shape of the tensegrity, only the pretension. We now examine the force-equilibrium conditions in more detail.

To begin, assume without any loss of generality that bar 12 is oriented such that $\boldsymbol{e}_{12}^T = [-1 \ 0]$ with $\{x_1 = y_1 = 0; x_2 > 0, y_2 = 0\}$. Then the first four rows of $\boldsymbol{A}$ in Eq. (20) are given by

$$\begin{bmatrix} \boldsymbol{H}_{12}\boldsymbol{e}_{13} & \boldsymbol{H}_{12}\boldsymbol{e}_{14} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{e}_{13} & \boldsymbol{e}_{14} & \boldsymbol{e}_{23} & \boldsymbol{e}_{24} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -y_3/\ell_{13} & -y_4/\ell_{14} & 0 & 0 \\ -x_3/\ell_{13} & -x_4/\ell_{14} & (x_2 - x_3)/\ell_{23} & (x_2 - x_4)/\ell_{24} \\ -y_3/\ell_{13} & -y_4/\ell_{14} & -y_3/\ell_{13} & -y_4/\ell_{24} \end{bmatrix} \tag{21}$$

All solutions $\boldsymbol{p} > \boldsymbol{0}$ of Eq. (19) are, therefore, of the form

$$\boldsymbol{p}^T = \beta[1 \quad p_2 \quad p_3 \quad p_4], \qquad \beta > 0$$

where

$$p_2 = -\left(\frac{\ell_{14}}{\ell_{13}}\right)\left(\frac{y_3}{y_4}\right), \qquad p_3 = -\left(\frac{\ell_{13}}{\ell_{24}}\right)\left(\frac{y_4}{y_3}\right)p_4$$

$$\left[\left(\frac{\ell_{13}}{\ell_{24}}\right)\left(\frac{y_4}{y_3}\right)\left(\frac{x_2 - x_3}{\ell_{23}}\right) - \left(\frac{x_2 - x_4}{\ell_{24}}\right)\right]p_4 = \left(\frac{x_4}{\ell_{13}}\right)\left(\frac{y_3}{y_4}\right) - \left(\frac{x_3}{\ell_{13}}\right)$$

Consequently, we have the following results.

Corollary: Consider the (2, 4; 2) tensegrity structure as illustrated in Fig. 9 where node k has coordinates (x_k, y_k) with $\{x_1 = y_1 = 0; x_2 > 0, y_2 = 0\}$, and the length of the cable $[k, m]$ is ℓ_{km}. Then the structure can be in force equilibrium with all positive force coefficients only if

$$y_3 y_4 < 0$$

$$\left[\left(\frac{\ell_{13}}{\ell_{24}}\right)\left|\frac{y_4}{y_3}\right|\left(\frac{x_2 - x_3}{\ell_{23}}\right) + \left(\frac{x_2 - x_4}{\ell_{24}}\right)\right]\left[\left(\frac{x_4}{\ell_{13}}\right)\left|\frac{y_3}{y_4}\right| + \left(\frac{x_3}{\ell_{13}}\right)\right] > 0 \tag{22}$$

In particular, the following conditions must be met:

1) Sufficient conditions for satisfaction of (3.22) are $y_3 y_4 < 0$ and $\{x_2 > x_4 > 0; x_2 \geq x_3 > 0\}$.

2) Force equilibrium can never exist for $y_3 y_4 > 0$ or $\{x_4 > x_2 > 0; x_3 > x_2 > 0\}$.

3) Given $y_3 y_4 < 0$ and $\{x_2 > x_4 > 0; x_3 \geq x_2 > 0\}$, a necessary condition for force equilibrium to exist is

$$|x_2 - x_3| < (\ell_{23}/\ell_{13})|y_3/y_4|(x_2 - x_4)$$

If we examine Fig. 9 (with bar 12 reoriented to comply with the assumptions in Corollary 1), we see that the condition $y_3 y_4 < 0$ implies that the nodes 3 and 4 can not be on the same side of bar 12, that is, the configuration in Fig. 9b can never define a (2, 4; 2) tensegrity structure in force equilibrium. This statement is a confirmation of the earlier observation that because the resultant forces $\boldsymbol{t}_{13}$ and $\boldsymbol{t}_{14}$ in Fig. 9b do not point along bar 12, then this geometrically consistent structure can not be in force equilibrium in the absence of external forces.

B. Single-Stage Structures

We now consider conditions for mechanical equilibrium for the $(N, 3N; N)$ tensegrity structure.

Lemma 2:

1) The geometry of an $(N, 3N; N)$ structure is determined by the unit vectors $\{\boldsymbol{e}_{kn}\}$ with cable and bar lengths $\{\ell_{pq} > 0, L_{2m-1,2m} > 0\}$ for all p, q, and m constrained as follows:

$$\ell_{1,2N}\boldsymbol{e}_{1,2N} - \ell_{2,2N}\boldsymbol{e}_{2,2N} = L_{1,2}\boldsymbol{e}_{1,2}, \qquad \ell_{1,3}\boldsymbol{e}_{1,3} - \ell_{2,3}\boldsymbol{e}_{2,3} = L_{1,2}\boldsymbol{e}_{1,2}$$

$$-\ell_{2N-2,2N-1}\boldsymbol{e}_{2N-2,2N-1} + \ell_{2N-2,2N}\boldsymbol{e}_{2N-2,2N} = L_{2N-1,2N}\boldsymbol{e}_{2N-1,2N}$$

$$-\ell_{1,2N-1}\boldsymbol{e}_{1,2N-1} + \ell_{1,2N}\boldsymbol{e}_{1,2N} = L_{2N-1,2N}\boldsymbol{e}_{2N-1,2N} \tag{23}$$

and, for $2 \leq m \leq N - 1$,

$$-\ell_{2m,2m+1}\boldsymbol{e}_{2m,2m+1} + \ell_{2m-1,2m+1}\boldsymbol{e}_{2m-1,2m+1} = L_{2m-1,2m}\boldsymbol{e}_{2m-1,2m}$$

$$-\ell_{2m-2,2m}\boldsymbol{e}_{2m-2,2m} - \ell_{2m-2,2m-1}\boldsymbol{e}_{2m-2,2m-1} = L_{2m-1,2m}\boldsymbol{e}_{2m-1,2m} \tag{24}$$

2) The resultant forces $\{\boldsymbol{t}_m\}$ are given by

$$\boldsymbol{t}_1 = \boldsymbol{t}_{13} + \boldsymbol{t}_{1,2N-1} + \boldsymbol{t}_{1,2N}, \qquad \boldsymbol{t}_2 = \boldsymbol{t}_{23} + \boldsymbol{t}_{2,4} + \boldsymbol{t}_{2,2N}$$

$$\boldsymbol{t}_{2N-1} = -\boldsymbol{t}_{2N-3,2N-1} - \boldsymbol{t}_{2N-2,2N-1} - \boldsymbol{t}_{1,2N-1}$$

$$\boldsymbol{t}_{2N} = -\boldsymbol{t}_{1,2N} - \boldsymbol{t}_{2,2N} - \boldsymbol{t}_{2N-2,2N} \tag{25}$$

and, for $2 \leq m \leq N - 1$, by

$$\boldsymbol{t}_{2m-1} = -\boldsymbol{t}_{2m-3,2m-1} - \boldsymbol{t}_{2m-2,2m-1} + \boldsymbol{t}_{2m-1,2m+1}$$

$$\boldsymbol{t}_{2m} = -\boldsymbol{t}_{2m-2,2m} + \boldsymbol{t}_{2m,2m+1} + \boldsymbol{t}_{2m,2m+2} \tag{26}$$

Definition 2: 1) Define the vectors $\boldsymbol{p}_1$ and $\boldsymbol{p}_{2N-1}$ by

$$\boldsymbol{p}_1^T = [\alpha_{13} \quad \alpha_{1,2N-1} \quad \alpha_{1,2N} \quad \alpha_{23} \quad \alpha_{24} \quad \alpha_{2,2N}]$$

$$\boldsymbol{p}_R^T = [\alpha_{1,R} \quad \alpha_{R-2,R} \quad \alpha_{R-1,R} \quad \alpha_{1,R+1} \quad \alpha_{2,R+1} \quad \alpha_{R-1,R+1}]$$

$$R = 2N - 1 \tag{27}$$

and the vectors $\{\boldsymbol{p}_n; n = 2m - 1; 2 \leq m \leq N - 1\}$ by

$$\boldsymbol{p}_n^T = [\alpha_{n-2,n} \quad \alpha_{n-1,n} \quad \alpha_{n,n+2} \quad \alpha_{n-1,n+1} \quad \alpha_{n+1,n+2} \quad \alpha_{n+1,n+3}] \tag{28}$$

2) Define the vectors $\boldsymbol{l}_1$ and $\boldsymbol{l}_{2N-1}$ by

$$\boldsymbol{l}_1^T = [\ell_{13} \quad \ell_{1,2N-1} \quad \ell_{1,2N} \quad \ell_{23} \quad \ell_{24} \quad \ell_{2,2N}]$$

$$\boldsymbol{l}_R^T = [\ell_{1,R} \quad \ell_{R-2,R} \quad \ell_{R-1,R} \quad \ell_{1,R+1} \quad \ell_{2,R+1} \quad \ell_{R-1,R+1}]$$

$$R = 2N - 1 \tag{29}$$

and the vectors $\{\boldsymbol{l}_n; n = 2m - 1; 2 \leq m \leq N - 1\}$ by

$$\boldsymbol{l}_n^T = [\ell_{n-2,n} \quad \ell_{n-1,n} \quad \ell_{n,n+2} \quad \ell_{n-1,n+1} \quad \ell_{n+1,n+2} \quad \ell_{n+1,n+3}] \tag{30}$$

From Eqs. (4), (6), (25), and (26), we then have the next result.

Lemma 3: Define the matrices $\boldsymbol{A}_1$ and $\{\boldsymbol{A}_P;\ P = 2N-1\}$ by

$$\boldsymbol{A}_1 = \begin{bmatrix} \boldsymbol{H}_{12}\boldsymbol{e}_{13} & \boldsymbol{H}_{12}\boldsymbol{e}_{1,2N-1} & \boldsymbol{H}_{12}\boldsymbol{e}_{1,2N} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{e}_{13} & \boldsymbol{e}_{1,2N-1} & \boldsymbol{e}_{1,2N} & \boldsymbol{e}_{23} & \boldsymbol{e}_{24} & \boldsymbol{e}_{2,2N} \end{bmatrix}$$

$$\boldsymbol{A}_P = \begin{bmatrix} \boldsymbol{H}_{P,P+1}\boldsymbol{e}_{1,P} & \boldsymbol{H}_{P,P+1}\boldsymbol{e}_{P-2,P} & \boldsymbol{H}_{P,P+1}\boldsymbol{e}_{P-1,P} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{e}_{1,P} & \boldsymbol{e}_{P-2,P} & \boldsymbol{e}_{P-1,P} & \boldsymbol{e}_{1,P+1} & \boldsymbol{e}_{2,P+1} & \boldsymbol{e}_{P-1,P+1} \end{bmatrix} \tag{31}$$

and $\{\boldsymbol{A}_k;\ k = 2m-1;\ 2 \le m \le N-1\}$ by

$$\boldsymbol{A}_k = \begin{bmatrix} \boldsymbol{H}_{k,k+1}\boldsymbol{e}_{k-2,k} & \boldsymbol{H}_{k,k+1}\boldsymbol{e}_{k-1,k} & -\boldsymbol{H}_{k,k+1}\boldsymbol{e}_{k,k+2} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{e}_{k-2,k} & \boldsymbol{e}_{k-1,k} & -\boldsymbol{e}_{k,k+2} & \boldsymbol{e}_{k-1,k+1} & -\boldsymbol{e}_{k+1,k+2} & -\boldsymbol{e}_{k+1,k+3} \end{bmatrix}$$

where $\boldsymbol{H}_{2m-1,2m}$ is defined by Eq. (5).

Then, a necessary condition for force equilibrium is the existence of a solution $\boldsymbol{p}_{2m-1}$ of the equation

$$\boldsymbol{A}_{2m-1}\boldsymbol{p}_{2m-1} = \boldsymbol{0}, \qquad \boldsymbol{p}_{2m-1} > \boldsymbol{0} \tag{32}$$

These conditions for force equilibrium are coupled because components of $\boldsymbol{p}_{2m-1}$ and $\boldsymbol{p}_{2n-1}$ for $m \neq n$ are not necessarily independent. The degree of independence is expressed in the following result.

Lemma 4:

1) The vectors $\{\boldsymbol{p}_{2r-1}, \boldsymbol{p}_{2s-1}\}$ have no components in common except when $\{r = 1, s = N\}$ and $\{r = m, s = m+1; 2 \le m \le N-1\}$.

2) The vectors $\{\boldsymbol{p}_1, \boldsymbol{p}_{2N-1}\}$ have components $\{\alpha_{1,2N-1}, \alpha_{1,2N}, \alpha_{2,2N}\}$ in common.

3) The vectors $\{\boldsymbol{p}_{2m-1}, \boldsymbol{p}_{2m+1}\}$ for $2 \le m \le N-1$ have components $\{\alpha_{2m-1,2m+1}, \alpha_{2m,2m+1}, \alpha_{2m,2m+2}\}$ in common.

This result motivates the following definition.

Definition 3:

1) Define the $6N$-dimensional vector $\tilde{\boldsymbol{p}}$ to be the concatenation of the six-dimensional vectors $\{\boldsymbol{p}_{2m-1}; 1 \le m \le N\}$ in Eqs. (27) and (28), that is,

$$\tilde{\boldsymbol{p}} = \begin{bmatrix} \boldsymbol{p}_1^T & \boldsymbol{p}_3^T & \cdots & \boldsymbol{p}_{2N-3}^T & \boldsymbol{p}_{2N-1}^T \end{bmatrix}^T \tag{33}$$

Then define the $3N$-dimensional cable coefficient vector $\boldsymbol{p}$ by progressing down the components of $\tilde{\boldsymbol{p}}$ and eliminating any component of $\tilde{\boldsymbol{p}}$ that already appears in $\boldsymbol{p}$.

2) Define the $6N$-dimensional vector $\tilde{\boldsymbol{l}}$ to be the concatenation of the six-dimensional vectors $\{\boldsymbol{l}_{2m-1}; 1 \le m \le N\}$ in Eqs. (29) and (30), that is,

$$\tilde{\boldsymbol{l}} = \begin{bmatrix} \boldsymbol{l}_1^T & \boldsymbol{l}_3^T & \cdots & \boldsymbol{l}_{2N-3}^T & \boldsymbol{l}_{2N-1}^T \end{bmatrix}^T \tag{34}$$

Then define the $3N$-dimensional cable length vector $\boldsymbol{l}$ by progressing down the components of $\tilde{\boldsymbol{l}}$ and eliminating any component of $\tilde{\boldsymbol{l}}$ that already appears in $\boldsymbol{l}$.

Note that the first six components in $\tilde{\boldsymbol{p}}$ (or $\tilde{\boldsymbol{l}}$) are given by the components of $\boldsymbol{p}_1$ (or $\boldsymbol{l}_1$), and that by the time we progress to $\boldsymbol{p}_{2N-1}$ (or $\boldsymbol{l}_{2N-1}$) in $\tilde{\boldsymbol{p}}$ (or $\tilde{\boldsymbol{l}}$), all components of $\boldsymbol{p}_{2N-1}$ (or $\boldsymbol{l}_{2N-1}$) will have already been included in $\boldsymbol{p}$ (or $\boldsymbol{l}$). For example, when $N = 4$, we have, from Eq. (33), that

$$\tilde{\boldsymbol{p}}^T = \begin{bmatrix} \boldsymbol{p}_1^T & \boldsymbol{p}_3^T & \boldsymbol{p}_5^T & \boldsymbol{p}_7^T \end{bmatrix}$$

where, from Eqs. (27) and (28)

$$\begin{aligned} \boldsymbol{p}_1^T &= [\alpha_{13} \quad \alpha_{17} \quad \alpha_{18} \quad \alpha_{23} \quad \alpha_{24} \quad \alpha_{28}] \\ \boldsymbol{p}_3^T &= [\alpha_{13} \quad \alpha_{23} \quad \alpha_{35} \quad \alpha_{24} \quad \alpha_{45} \quad \alpha_{46}] \\ \boldsymbol{p}_5^T &= [\alpha_{35} \quad \alpha_{45} \quad \alpha_{57} \quad \alpha_{46} \quad \alpha_{67} \quad \alpha_{68}] \\ \boldsymbol{p}_7^T &= [\alpha_{17} \quad \alpha_{57} \quad \alpha_{67} \quad \alpha_{18} \quad \alpha_{28} \quad \alpha_{68}] \end{aligned}$$

Then, the corresponding ($3N$=) 12-dimensional vector $\boldsymbol{p}$ is given by

$$\boldsymbol{p}^T = \begin{bmatrix} \boldsymbol{p}_1^T, & \alpha_{35}, & \alpha_{45}, & \alpha_{46}, & \alpha_{57}, & \alpha_{67}, & \alpha_{68} \end{bmatrix} \tag{35}$$

We now have the following result.

Theorem 5: Consider a $(N, 3N; N)$ tensegrity structure in which the $2N$ endpoints have coordinates $\{[x_m \ y_m \ z_m]; m = 1, 2, \ldots, 2N\}$. Define the cable length vector $\boldsymbol{l}$ and the cable coefficient vector $\boldsymbol{p}$ as in Definition 3. Then the following hold:

1) The geometry determined by the unit vectors $\{\boldsymbol{e}_{kn}\}$ is consistent if there exists a solution $\boldsymbol{l}$ of the equation

$$\boldsymbol{G}\boldsymbol{l} = \boldsymbol{b}, \qquad \boldsymbol{l} > \boldsymbol{0} \tag{36}$$

where

$$\boldsymbol{b} = \begin{bmatrix} L_{12}\boldsymbol{e}_{12}^T & L_{12}\boldsymbol{e}_{12}^T & \ldots & L_{2N-1,2N}\boldsymbol{e}_{2N-1,2N}^T & L_{2N-1,2N}\boldsymbol{e}_{2N-1,2N}^T \end{bmatrix}^T$$

for some $6N \times 3N$ matrix $\boldsymbol{G}$ defined from Eqs. (23) and (24), which is a function of only the cable unit vectors $\{\boldsymbol{e}_{kn}\}$.

2) The consistent geometry defined by Eq. (36) is in force equilibrium if and only if there exists a vector $\boldsymbol{p}$ of the equation

$$\boldsymbol{A}\boldsymbol{p} = \boldsymbol{0}, \qquad \boldsymbol{p} > \boldsymbol{0} \tag{37}$$

for some $6N \times 3N$ matrix $\boldsymbol{A}$ defined from Eq. (32) for $m = 1, 2, \ldots, N$, which is a function of only the cable and bar unit vectors $\{\boldsymbol{e}_{kn}\}$.

For example, consider the (3, 9; 3) tensegrity illustrated in Fig. 2. Then $N = 3$ and the ($3N$=) 9-dimensional vector $\boldsymbol{p}$ is given by

$$\boldsymbol{p} = [\alpha_{13} \quad \alpha_{15} \quad \alpha_{16} \quad \alpha_{23} \quad \alpha_{24} \quad \alpha_{26} \quad \alpha_{35} \quad \alpha_{45} \quad \alpha_{46}]^T$$

the corresponding 18×9 matrix $\boldsymbol{G}$ in Eq. (36) is given by

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{e}_{16} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{e}_{26} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{e}_{13} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{e}_{23} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{e}_{23} & \boldsymbol{e}_{24} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{e}_{35} & -\boldsymbol{e}_{45} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{e}_{45} & \boldsymbol{e}_{46} \\ \boldsymbol{0} & -\boldsymbol{e}_{15} & \boldsymbol{e}_{16} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$$

and the corresponding 18×9 matrix $\boldsymbol{A}$ in Eq. (37) is given by

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{H}_{12}\boldsymbol{e}_{13} & \boldsymbol{H}_{12}\boldsymbol{e}_{15} & \boldsymbol{H}_{12}\boldsymbol{e}_{16} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{e}_{13} & \boldsymbol{e}_{15} & \boldsymbol{e}_{16} & \boldsymbol{e}_{23} & \boldsymbol{e}_{24} & \boldsymbol{e}_{26} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{H}_{34}\boldsymbol{e}_{13} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{H}_{34}\boldsymbol{e}_{23} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{H}_{34}\boldsymbol{e}_{35} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{e}_{13} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{e}_{23} & \boldsymbol{e}_{24} & \boldsymbol{0} & -\boldsymbol{e}_{35} & -\boldsymbol{e}_{45} & -\boldsymbol{e}_{46} \\ \boldsymbol{0} & \boldsymbol{H}_{56}\boldsymbol{e}_{15} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{H}_{56}\boldsymbol{e}_{35} & \boldsymbol{H}_{56}\boldsymbol{e}_{45} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{e}_{15} & \boldsymbol{e}_{16} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{e}_{26} & \boldsymbol{e}_{35} & \boldsymbol{e}_{45} & \boldsymbol{e}_{46} \end{bmatrix}$$

The algebraic results presented in Theorem 5 for the one-stage $(N, 3N; N)$ tensegrity structure can be extended to the case of both M-stage symmetrical and nonsymmetrical structures based on the corresponding cable length vector $\boldsymbol{l}$ and the cable coefficient vector $\boldsymbol{p}$.

IV. Conclusions

This paper has defined the subclass $(N, S; P_1, P_2, \ldots, P_M)$ of class 1 tensegrity systems that (for $N \geq 3$) was defined topologically as a three-dimensional closure of a two-dimensional lattice configuration. The existence as a valid tensegrity system was inferred by the construction procedure. This subclass is characterized by N compressive elements (called bars), S tensile elements (called cables), and M stages with P_k bars per stage. The bars form a discontinuous network in compression, whereas the cables form a continuous network in tension. The Needle Tower sculpture of Snelson (which has inspired the last 50 years of interest in tensegrity systems) is a symmetrical $(N = MP, S = 4MP; P, P, \ldots, P)$ structure with M stages.

The subclass $(N, S; P_1, P_2, \ldots, P_M)$ includes both symmetrical and nonsymmetrical structures, but the permitted number of bars P_k in stage k can only either increase or decrease by one with respect to the number of bars P_{k-1} in the preceding stage. Specific results quantified this constraint, and established that the number of cables S required for the structure averaged between three and four per node. The minimum number of three was required for a single-stage structure, whereas the maximum number of four was required for a symmetrical $M(>1)$-stage structure.

Mechanical equilibrium of a tensegrity structure required both geometrical consistency and force-equilibrium conditions to be satisfied. Geometrical consistency was shown to be equivalent to finding a solution $\boldsymbol{l} > \mathbf{0}$ of a system of equations $\boldsymbol{G}\boldsymbol{l} = \boldsymbol{b}$, where $\boldsymbol{G}$ is a rectangular matrix that is a function of the unit vectors of the cables, and $\boldsymbol{b}$ is a vector that is a function of the unit vectors and the lengths of the bars. Given such a solution $\boldsymbol{l} > \mathbf{0}$, force equilibrium was shown to be equivalent to finding a solution $\boldsymbol{p} > \mathbf{0}$ of a system of equations $\boldsymbol{A}\boldsymbol{p} = \mathbf{0}$, where $\boldsymbol{A}$ is a rectangular matrix that is a function of the unit vectors of the cables and bars. Hence, not only must $\boldsymbol{A}$ have a nontrivial null space, but, in addition, there must be a vector in this null space that has all positive components. This latter condition is difficult to establish from both an analytical and a computational point of view, and more explicit necessary and sufficient conditions for mechanical equilibrium are under investigation.

The dimension of the null space of $\boldsymbol{A}$ for a tensegrity in mechanical equilibrium is also of interest because this property has implications on the conditions for mechanical prestress. If the null space of $\boldsymbol{A}$ has dimension one, then all tensions can only be scaled in proportion by adjusting α. However, if the null space of $\boldsymbol{A}$ has dimension greater than one, then other tension patterns are possible. For example, one may be interested in choosing a solution $\boldsymbol{p} > \mathbf{0}$ such that the stiffness characteristics of the structure in a particular direction are maximized subject to a given geometry and given cable spring constants. The stiffness characteristics of tensegrity structures are under investigation.

Acknowledgments

The assistance of Drew Whitehouse of the Australian National University Supercomputer Facility in writing the code and creating the visualization of a number of tensegrity structures is gratefully acknowledged. For visualizations, visit "Visuality of Tensegrity Structures," URL: http://anusf.anu.edu.au/Vizlab/viz_showcase/williamson_darrell/.

References

[1]Fuller, R. B., "Tensile-Integrity Structures," U.S. Patent 3,063,521, 1962.

[2]Fuller, R. B., and Marks, R., *The Dymaxion World of Buchminister Fuller*, Archor Books, 1973.

[3]Skelton, R. E., Helton, W. J., and Adhikari, R., "Mechanics of Tensegrity Beams," Structural Systems and Control Lab., Rept. 1998-1, Univ. of California, San Diego, La Jolla, CA, March 1998.

[4]Connelly, R., and Back, A., "Mathematics and Tensegrity," *American Scientist*, Vol. 86, March 1998.

[5]Ingber, D. E., "Tensegrity: The Architectural Basis of Cellular Mechanotransduction," *Annual Review of Physiology*, Vol. 59, 1997, pp. 575–599.

[6]Ingber, D. E., "Architecture of Life," *Scientific American*, Jan. 1998, pp. 48–57.

[7]Furuya, H., "Concept of Deployable Tensegrity Structures in Space Applications," *International Journal of Space Structures*, Vol. 7, No. 2, 1992, pp. 143–151.

[8]Motro, R., "Tensegrity Systems: The State of the Art," *International Journal of Space Structures*, Vol. 7, No. 2, 1992, pp. 75–83.

[9]Skelton, R. E., and Sultan, C., "Controllable Tensegrity," Proc. SPIE 4th Ann. Symp. Smart Str. and Mats., March 1997.

[10]Sultan, C., Corless, M., and Skelton, R. E., "Reduced Prestressability Conditions for Tensegrity Structures," AIAA, April 1999.

[11]Williamson, D., and Skelton, R. E., "A General Class of Tensegrity Systems: Geometric Definition," American Society of Civil Engineers, May 1998.

[12]Skelton, R. E., Pinaud, J. P., and Mingori, D. I., "Dynamics of the Shell Class of Tensegrity Structures," *Journal of the Franklin Institute*, Vol. 338, 2001, pp. 255–320.